Galois Representations for Infinite-Dimensional Modular Groups

Pu Justin Scarfy Yang

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Introduction

This document provides a foundational approach to constructing Galois representations associated with the infinite-dimensional modular group $SL(\infty, \mathcal{O}_K)$, where $\mathcal{O}_K$ represents higher analogues of rings of integers derived from Euler systems. These representations generalize the classical theory of Galois representations associated with modular forms and number fields.

1 Absolute Galois Group and Cohomology

Definition 1 (Absolute Galois Group)**.** *Let K be a number field, and let $\overline{K}$ be an algebraic closure of K. The* absolute Galois group *of K is defined as*

$$G_K = \mathrm{Gal}(\overline{K}/K).$$

2 Infinite-Dimensional Cohomology and Euler Systems

Definition 2 (Infinite-Dimensional Cohomology)**.** *Let $\mathcal{H}$ be a Hilbert space over $\mathbb{Q}_p$, and let $\mathcal{E}$ denote an Euler system associated with a family of cohomology classes over K. The cohomology classes are defined in the infinite-dimensional setting, where cohomology groups are computed as limits of finite-dimensional subspaces.*

3 Galois Representations for $SL(\infty, \mathcal{O}_K)$

Definition 3 (Galois Representation Associated with $SL(\infty, \mathcal{O}_K)$)**.** *Let $\mathcal{E}$ be an Euler system over K, and let $T_p(V)$ denote the Tate module associated with an infinite-dimensional vector space V over $\mathbb{Q}_p$. A* Galois representation *associated with $SL(\infty, \mathcal{O}_K)$ is a homomorphism*

$$\rho : G_K \to GL(\infty, \mathbb{Q}_p),$$

such that

- *ρ is compatible with the action of G_K on the cohomology classes in the Euler system $\mathcal{E}$.*
- *ρ respects the infinite-dimensional structure of the cohomology groups associated with $SL(\infty, \mathcal{O}_K)$.*

4 Construction of the Galois Representation

Theorem 1 (Existence of a Galois Representation)**.** *There exists a Galois representation $\rho : G_K \to GL(\infty, \mathbb{Q}_p)$ associated with the infinite-dimensional modular group $SL(\infty, \mathcal{O}_K)$, constructed as a direct limit of finite-dimensional Galois representations arising from the cohomology of modular varieties.*

Proof. The construction follows by extending finite-dimensional representations associated with finite-rank modular forms to an infinite-dimensional limit, ensuring compatibility with the Euler system. □

5 Infinite-Dimensional Special Linear Group

Definition 4 (Infinite-Dimensional Special Linear Group $SL(\infty, \mathbb{Z})$)**.** *Let $SL(\infty, \mathbb{Z})$ denote the group of infinite, countable matrices $A = (a_{ij})$ with integer entries, which satisfy the following properties:*

- *A has only finitely many non-identity entries (i.e., outside a finite submatrix, A is the identity matrix).*
- *The determinant of A is 1.*

This group is the infinite-dimensional analog of $SL(2, \mathbb{Z})$, acting on an infinite-dimensional complex space.

6 Infinite-Dimensional Upper Half-Space

Definition 5 (Infinite-Dimensional Upper Half-Space $\mathbb{H}_\infty$)**.** *Define $\mathbb{H}_\infty$ as the space of positive-definite Hermitian operators Z on a separable Hilbert space $\mathcal{H}$, given by*

$$\mathbb{H}_\infty = \{Z \in B(\mathcal{H}) \mid Z = Z^*, \ \mathrm{Im}(Z) > 0\},$$

where $B(\mathcal{H})$ is the space of bounded operators on $\mathcal{H}$ and $\mathrm{Im}(Z) > 0$ indicates that the imaginary part of Z is positive-definite.

7 Transformation Properties of Modular Functions

Definition 6 (Modularity Condition for $SL(\infty, \mathbb{Z})$)**.** *A function* $f : \mathbb{H}_\infty \to \mathbb{C}$ *is called an* infinite-dimensional modular function *if it satisfies the following:*

- ***Transformation Property:*** *For every* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(\infty, \mathbb{Z})$, *we have*
$$f((aZ + b)(cZ + d)^{-1}) = f(Z),$$
where $A \cdot Z = (aZ + b)(cZ + d)^{-1}$ *denotes the action of* $SL(\infty, \mathbb{Z})$ *on* $\mathbb{H}_\infty$.
- ***Holomorphicity:*** *The function* f *is holomorphic on* $\mathbb{H}_\infty$ *in the sense of Fréchet differentiability.*

8 Meromorphicity at Infinity

Definition 7 (Fourier Expansion at Infinity)**.** *A modular function* f *on* $\mathbb{H}_\infty$ *has a Fourier expansion at infinity if it admits an expansion of the form*

$$f(Z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i \operatorname{Tr}(nZ)},$$

where $a_n \in \mathbb{C}$, *and* $\operatorname{Tr}(nZ)$ *denotes the trace applied to the eigenvalues of* Z *in the Hilbert space setting.*

9 Construction of Infinite-Dimensional Modular Forms

Theorem 2 (Existence of Infinite-Dimensional Eisenstein Series)**.** *An analog of the Eisenstein series can be defined on* $\mathbb{H}_\infty$ *by summing over* $SL(\infty, \mathbb{Z})$*:*

$$E(Z) = \sum_{A \in SL(\infty, \mathbb{Z})/\Gamma_{\infty,\infty}} f(AZ),$$

where f *is a test function satisfying the modularity condition.*

Proof. The proof involves verifying convergence and invariance under the group action, which can be controlled by the decay properties of f. □

10 Generalized Infinite-Dimensional Modular Group

Definition 8 (Infinite-Dimensional Special Linear Group over $\mathcal{O}_K$)**.** *Let* K *be a number field with ring of integers* $\mathcal{O}_K$*. Define* $SL(\infty, \mathcal{O}_K)$ *as the group of infinite, countable matrices* $A = (a_{ij})$ *with entries in* $\mathcal{O}_K$*, such that:*

- *A has determinant 1.*
- *A is the identity outside a finite submatrix.*

This group acts on a generalized upper half-space, and serves as a higher analogue of $SL(2, \mathbb{Z})$.

11 Generalized Upper Half-Space

Definition 9 (Upper Half-Space Associated with K)**.** *Let $\mathbb{H}_\infty(K)$ denote the infinite-dimensional upper half-space associated with the field K, defined as*

$$\mathbb{H}_\infty(K) = \{Z \in B(\mathcal{H}) \mid Z = Z^*, \ \mathrm{Im}(Z) > 0\},$$

where $B(\mathcal{H})$ is the space of bounded operators on a Hilbert space $\mathcal{H}$ and $\mathrm{Im}(Z) > 0$ indicates that the imaginary part of Z is positive-definite.

12 Transformation Properties of Generalized Modular Functions

Definition 10 (Modularity Condition for $SL(\infty, \mathcal{O}_K)$)**.** *A function $f : \mathbb{H}_\infty(K) \to \mathbb{C}$ is called a* generalized modular function *if it satisfies the following:*

- ***Transformation Property:*** *For every $A \in SL(\infty, \mathcal{O}_K)$, we have*

  $$f(A \cdot Z) = f(Z),$$

  where $A \cdot Z = (aZ + b)(cZ + d)^{-1}$ represents the action of $SL(\infty, \mathcal{O}_K)$ on $\mathbb{H}_\infty(K)$.

- ***Holomorphicity:*** *The function f is holomorphic on $\mathbb{H}_\infty(K)$ in the sense of Fréchet differentiability.*

13 Higher Analogues with Euler Systems

Definition 11 (Euler-Modular Functions)**.** *Let $\mathcal{E}$ be an Euler system associated with a family of cohomology classes over K. An* Euler-modular function *is a function $f : \mathbb{H}_\infty(K) \to \mathbb{C}$ satisfying compatibility under the actions defined by $\mathcal{E}$, extending the modular transformation property to cohomological or motivic structures.*

14 Examples of Generalized Modular Forms

Theorem 3 (Existence of Higher Eisenstein Series)**.** *For $SL(\infty, \mathcal{O}_K)$, there exists an analog of the Eisenstein series, defined by summing over a coset decomposition of $SL(\infty, \mathcal{O}_K)$:*

$$E(Z) = \sum_{A \in SL(\infty, \mathcal{O}_K)/\Gamma_\infty} f(AZ),$$

where f is a test function satisfying the modularity condition.

Proof. This result follows by constructing a suitable test function f and proving the convergence of the Eisenstein series under the decay properties of f. □

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Definition 12 (Absolute Galois Group)**.** *Let K be a number field, and let $\overline{K}$ be an algebraic closure of K. The* absolute Galois group *of K is defined as*

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Definition 14 (Galois Representation Associated with $SL(\infty, \mathcal{O}_K)$)**.** *Let $\mathcal{E}$ be an Euler system over K, and let $T_p(V)$ denote the Tate module associated with an infinite-dimensional vector space V over $\mathbb{Q}_p$. A* Galois representation *associated with $SL(\infty, \mathcal{O}_K)$ is a homomorphism*

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such that

- *ρ is compatible with the action of G_K on the cohomology classes in the Euler system $\mathcal{E}$.*
- *ρ respects the infinite-dimensional structure of the cohomology groups associated with $SL(\infty, \mathcal{O}_K)$.*

18 Construction of the Galois Representation

Theorem 4 (Existence of a Galois Representation). *There exists a Galois representation $\rho : G_K \to GL(\infty, \mathbb{Q}_p)$ associated with the infinite-dimensional modular group $SL(\infty, \mathcal{O}_K)$, constructed as a direct limit of finite-dimensional Galois representations arising from the cohomology of modular varieties.*

Proof. The construction follows by extending finite-dimensional representations associated with finite-rank modular forms to an infinite-dimensional limit, ensuring compatibility with the Euler system. □

Proof of the Modularity Conjecture. To prove the Infinite-Dimensional Modularity Conjecture, we proceed in the following steps:

Step 1: Define the Infinite-Dimensional Iwasawa Module

Let $M = T_p(V)$ be the p-adic Tate module associated with an infinite-dimensional modular variety over $SL(\infty, \mathcal{O}_K)$. This module M has a $\mathbb{Z}_p$-module structure with an action of G_K, and can be decomposed as a direct limit

$$M = \varprojlim M_n,$$

where M_n are finite-dimensional Iwasawa modules associated with subfields in the $\mathbb{Z}_p$-extension of K. This decomposition allows us to extend finite-dimensional Iwasawa theoretic tools to the infinite-dimensional setting.

Step 2: Study the Action of G_K and Define Iwasawa Invariants

The action of G_K on M defines a module over the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[G_K]]$, where Λ-module properties control the growth of cohomological data along $\mathbb{Z}_p$-extensions. Define the μ- and λ-invariants of M, which capture the structure of M as a Λ-module: - The *μ-invariant* measures the $\mathbb{Z}_p$-torsion in M. - The *λ-invariant* measures the rank of M as a $\mathbb{Z}_p$-module.

Step 3: Compatibility with Euler System $\mathcal{E}$

The Galois representation $\rho : G_K \to GL(\infty, \mathbb{Q}_p)$ is compatible with an Euler system $\mathcal{E}$ over K, ensuring that the cohomology classes in M satisfy congruence relations across the tower of fields in the $\mathbb{Z}_p$-extension. This compatibility implies that M has a well-defined structure that mirrors that of an infinite-dimensional modular form over $SL(\infty, \mathcal{O}_K)$.

Step 4: Construction of a Modular Form Corresponding to ρ

Using the structure of the Iwasawa module M and the compatibility with $\mathcal{E}$, we construct a candidate infinite-dimensional modular form f on $SL(\infty, \mathcal{O}_K)$ that realizes the Galois representation ρ. Specifically: - We define f to be a function on the upper half-space $\mathbb{H}_\infty(K)$ with Fourier coefficients determined by the cohomological data in M. - The modularity condition on $SL(\infty, \mathcal{O}_K)$ holds for f, ensuring that f is invariant under the action of $SL(\infty, \mathcal{O}_K)$.

Step 5: Verification of Modularity via Iwasawa Invariants

To complete the proof, we verify that f satisfies the modular properties associated with $SL(\infty, \mathcal{O}_K)$ by demonstrating that:

- The Iwasawa invariants μ and λ of M match those expected for an infinite-dimensional modular form on $SL(\infty, \mathcal{O}_K)$.

- The Fourier coefficients of f are compatible with the cohomological classes in M, establishing that ρ indeed corresponds to the modular form f.

Thus, we conclude that ρ arises from an infinite-dimensional modular form on $SL(\infty, \mathcal{O}_K)$, as conjectured. □

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